

Strong coupling in massive gravity by direct calculation

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Abstract

We consider four-dimensional massive gravity with the Fierz–Pauli mass term. The analysis of the scalar sector has revealed recently that this theory becomes strongly coupled above the energy scale $\Lambda = (M_P m^4)^{\frac{1}{5}}$ where m is the mass of the graviton. We confirm this scale by explicit calculations of the four-graviton scattering amplitude and of the loop correction to the interaction between conserved sources.

1 Introduction and summary

In view of the evidence for the accelerated expansion of the Universe, it is of interest to understand whether there exist consistent and phenomenologically acceptable gravitational theories which deviate from general relativity at cosmological scales. The simplest possibility would be to give a mass to the graviton by adding the Fierz–Pauli term [1] to the Einstein action. At the linearized level, this would modify classical gravity even at energy scales exceeding the graviton mass, due to the van Dam–Veltman–Zakharov phenomenon [2], but this undesirable feature may be cured by non-linear effects [3, 4]. More serious problem emerges at quantum level: it has been argued [5] that the theory becomes strongly coupled above the energy scale

$$\Lambda = (M_P m^4)^{\frac{1}{5}} \quad (1)$$

where m is the graviton mass. For $m \sim H_0$, the present value of the Hubble parameter, this scale is unacceptably low. Possible higher order terms do not improve the ultraviolet behaviour to phenomenologically acceptable level [5]. Similar problem is inherent also in brane-world models [6, 7, 8]

with gravity modified at ultra-large distances: they either are strongly coupled at unacceptably low energies [9, 10, 11, 12] or have ghosts [13, 9, 11, 12]. A possible way out is related to the breaking of the Lorentz invariance, as suggested recently [14].

The energy scale (1) was derived in Ref. [5] in a somewhat indirect way, namely, by making use of a gravitational analogue of the sigma-model approach. It is of interest to see this scale directly in scattering amplitudes and loop corrections. This is precisely the purpose of this paper: we calculate directly the $2 \rightarrow 2$ scattering amplitude of longitudinal gravitons (Figs. 1 and 2) and the loop correction to the graviton propagator (Fig. 3), both

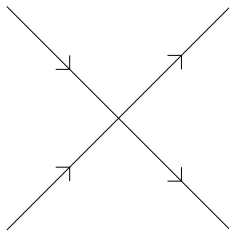


Figure 1: Scattering through the four point vertex.

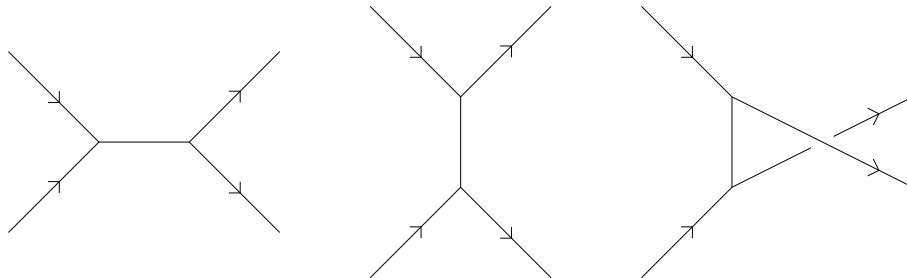


Figure 2: Scattering via the s-, t- and u-channel exchange.

in flat background. We confirm that the strong coupling scale in massive gravity is given by Eq. (1). This scale appears in the two calculations in a way which is not completely trivial: a naïve power counting would suggest even lower energy scale, but two leading orders in m^{-1} cancel out in both cases. As an example, the wave function of the longitudinal graviton is proportional to m^{-2} , while the largest term in the graviton propagator is proportional to m^{-4} . Thus, naïve power counting would suggest that the diagrams of Fig. 2 are of the order $\frac{E^{14}}{M_{Pl}^2 m^{12}}$. However, on-shell amplitude is in fact of order $\frac{E^{10}}{M_{Pl}^2 m^8}$, due to cancellations. This immediately implies that the strong interaction scale is indeed given by Eq. (1).

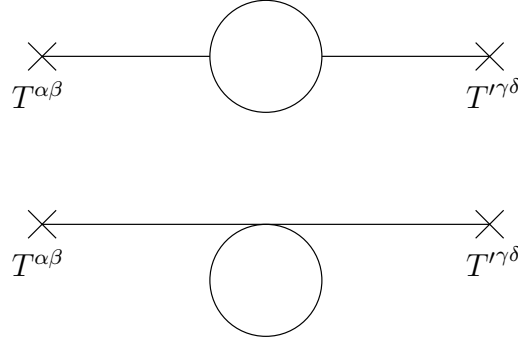


Figure 3: Graviton propagator at one loop.

2 Massive graviton wave functions and propagator

The graviton field $h_{\mu\nu}(x)$ is the perturbation about the Minkowski metric $\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$,

$$h_{\mu\nu}(x) = g_{\mu\nu}(x) - \eta_{\mu\nu} \quad (2)$$

The graviton is given a mass by adding the Fierz-Pauli term [1] to the Einstein action,

$$S = -\frac{M_{Pl}^2}{16\pi} \int d^4x \sqrt{-g} \mathcal{R} - f^4 \int d^4x \left(h_{\mu\nu} h^{\mu\nu} - h^\mu{}_\mu h^\nu{}_\nu \right) \quad (3)$$

where the indices of $h_{\mu\nu}(x)$ are raised by the Minkowski metric.

By solving the linearised field equation, one finds that the graviton field has a mass, $m^2 = \frac{64\pi f^4}{M_{Pl}^2}$, and is decomposed as follows,

$$h_{\mu\nu}(x) = \frac{4\sqrt{2\pi}}{M_{Pl}} \int \frac{d^3k}{(2\pi)^3(2k^0)} \sum_{\alpha=1}^5 \left(e_{\mu\nu}^{(\alpha)} a^{(\alpha)}(\mathbf{k}) e^{-ikx} + h.c. \right) \quad (4)$$

where the five polarisation tensors have the following proprieties,

$$e_{\mu\nu}^{(\alpha)} = e_{\nu\mu}^{(\alpha)} \quad ; \quad e^{(\alpha)\mu}{}_\mu = 0 \quad ; \quad k_\mu e^{(\alpha)\mu\nu} = 0 \quad ; \quad e_{\mu\nu}^{(\alpha)} e^{(\beta)\mu\nu} = \delta^{\alpha\beta}$$

and the creation and annihilation operators are normalized in the standard way,

$$[a^{(\alpha)}(\mathbf{k}), a^{\beta\dagger}(\mathbf{k}')] = (2\pi)^3 (2k^0) \delta^3(\mathbf{k} - \mathbf{k}') \delta^{\alpha\beta}$$

The five polarisation tensors $e_{\mu\nu}^{(\alpha)}$ of the massive graviton can be expressed in terms of the three polarisation vectors $e_\mu^{(i)}$ of the massive vector

field,

$$\begin{aligned}
e_{\mu\nu}^{(1)} &= \frac{1}{\sqrt{2}}(e_\mu^{(1)}e_\nu^{(2)} + e_\mu^{(2)}e_\nu^{(1)}) & ; & & e_{\mu\nu}^{(2)} &= \frac{1}{\sqrt{2}}(e_\mu^{(1)}e_\nu^{(1)} - e_\mu^{(2)}e_\nu^{(2)}) \\
e_{\mu\nu}^{(3)} &= \frac{1}{\sqrt{2}}(e_\mu^{(1)}e_\nu^{(3)} + e_\mu^{(3)}e_\nu^{(1)}) & ; & & e_{\mu\nu}^{(4)} &= \frac{1}{\sqrt{2}}(e_\mu^{(2)}e_\nu^{(3)} + e_\mu^{(3)}e_\nu^{(2)}) \\
e_{\mu\nu}^{(5)} &= \frac{1}{\sqrt{6}}(e_\mu^{(1)}e_\nu^{(1)} + e_\mu^{(2)}e_\nu^{(2)} - 2e_\mu^{(3)}e_\nu^{(3)}) & & & &
\end{aligned} \tag{5}$$

where the two first polarisation tensors (as well as the two first polarisation vectors) are the ones of the massless case.

In a frame where $k^\mu = (k^0, 0, 0, k^3)$, one has

$$e_\mu^{(1)} = (0, 1, 0, 0) \quad ; \quad e_\mu^{(2)} = (0, 0, 1, 0) \quad ; \quad e_\mu^{(3)} = \frac{1}{m} (k^3, 0, 0, -k^0)$$

Hence, the longitudinal tensor $e_{\mu\nu}^{(5)}$ is the most singular one as $m \rightarrow 0$. For further calculations it is convenient to express it in terms of momentum,

$$e_{\mu\nu}^{(5)} = -\sqrt{\frac{2}{3}} \frac{k_\mu k_\nu}{m^2} + \mathcal{O}(m^0) \tag{6}$$

The graviton propagator obeys the following equation,

$$\begin{aligned}
&\frac{M_{Pl}^2}{64\pi} \left[(\eta_{\gamma\alpha}\eta_{\delta\beta} + \eta_{\gamma\beta}\eta_{\delta\alpha} - 2\eta_{\gamma\delta}\eta_{\alpha\beta})(k^2 - m^2) \right. \\
&\quad \left. - (k_\delta\eta_{\gamma\alpha}k_\beta + k_\delta\eta_{\gamma\beta}k_\alpha + k_\gamma\eta_{\delta\alpha}k_\beta + k_\gamma\eta_{\delta\beta}k_\alpha) \right. \\
&\quad \left. + 2(\eta_{\gamma\delta}k_\alpha k_\beta + k_\gamma k_\delta \eta_{\alpha\beta}) \right] G^{\alpha\beta\mu\nu}(k) = \frac{i}{2} (\delta_\gamma^\mu \delta_\delta^\nu + \delta_\gamma^\nu \delta_\delta^\mu)
\end{aligned}$$

One finds that the massive graviton propagator contains terms of order 1, m^{-2} and m^{-4} ,

$$\begin{aligned}
G_{\alpha\beta\mu\nu}(k) &= \frac{16\pi i}{M_{Pl}^2(k^2 - m^2)} [\eta_{\alpha\mu}\eta_{\beta\nu} + \eta_{\alpha\nu}\eta_{\beta\mu} - \frac{2}{3}\eta_{\alpha\beta}\eta_{\mu\nu} \\
&\quad - \frac{1}{m^2}(k_\beta\eta_{\alpha\mu}k_\nu + k_\beta\eta_{\alpha\nu}k_\mu + k_\alpha\eta_{\beta\mu}k_\nu + k_\alpha\eta_{\beta\nu}k_\mu) \\
&\quad + \frac{2}{3m^2}(\eta_{\alpha\beta}k_\mu k_\nu + k_\alpha k_\beta \eta_{\mu\nu}) + \frac{4}{3m^4}k_\alpha k_\beta k_\mu k_\nu] \tag{7}
\end{aligned}$$

The propagator can also be written as follows,

$$G_{\alpha\beta\mu\nu}(k) = \frac{16\pi i}{M_{Pl}^2(k^2 - m^2)} [\hat{\eta}_{\alpha\mu}\hat{\eta}_{\beta\nu} + \hat{\eta}_{\alpha\nu}\hat{\eta}_{\beta\mu} - \frac{2}{3}\hat{\eta}_{\alpha\beta}\hat{\eta}_{\mu\nu}] \tag{8}$$

where $\hat{\eta}_{\mu\nu} = \eta_{\mu\nu} - \frac{k_\mu k_\nu}{m^2}$. The coefficient $-\frac{2}{3}$ here is different from -1 of the massless case; this is precisely the van Dam–Veltman–Zakharov discontinuity [2].

As a cross check, near mass shell one has

$$G_{\alpha\beta\mu\nu}(k) = \frac{32\pi i}{M_{Pl}^2(k^2 - m^2)} \sum_{\gamma=1}^5 e_{\alpha\beta}^{(\gamma)} e_{\mu\nu}^{(\gamma)}$$

which is precisely the expected contribution of the nearly on-shell gravitons.

3 Scattering

Interaction between gravitons comes from non-linear terms in the Einstein action. The corresponding three-point and four-point vertices are given in Appendix. We use them for calculating the amplitude of scattering of two massive gravitons; at the tree level it is given by the sum of the diagrams shown in Figs. 1 and 2.

By naïve power counting, one might think that the term of order m^{-4} in the massive propagator (7) dominates the scattering amplitude. However, once the explicit form of the vertex (16) is used, and external legs are taken on-shell, its contribution in fact picks up a factor m^4 . Likewise, the term in the propagator with m^{-2} in front picks up a factor m^2 , so effectively the propagator is of order m^0 . These cancellations occur for each diagram in Fig. 2 separately, and for all polarisations of gravitons in the legs, provided these are on-shell. Then, the inverse powers of the mass m in the scattering amplitude come only from the polarisation tensor $e_{\mu\nu}^{(\alpha)}$.

Therefore, the largest amplitude at $E \gg m$ involves longitudinal gravitons in all four legs. Making use of Eq. (6), we obtain for the diagram of Fig. 1, the diagram of Fig. 2 and their sum:

$$\mathcal{M}_4 = -\frac{2\pi}{9M_{Pl}^2 m^8} stu(s^2 + t^2 + u^2) \quad (9)$$

$$\mathcal{M}_3 = \frac{7\pi}{54M_{Pl}^2 m^8} stu(s^2 + t^2 + u^2) \quad (10)$$

$$\mathcal{M}_{TOT} = -\frac{5\pi}{54M_{Pl}^2 m^8} stu(s^2 + t^2 + u^2) \quad (11)$$

where s , t and u are the Mandelstam variables. These expressions are symmetric in s , t and u as they should.

Thus, at high energies the $2 \rightarrow 2$ amplitude for longitudinal gravitons is of the order

$$\mathcal{M} \sim \frac{E^{10}}{\Lambda^{10}}$$

where Λ is precisely the scale (1). The standard unitarity argument implies then that the theory is strongly coupled at $E \gtrsim \Lambda$, in agreement with Ref. [5].

4 One-loop graviton propagator

Another way to see the strong coupling scale is to study the loop correction to the graviton propagator. We do this by adding external sources of the form $T^{\alpha\beta}h_{\alpha\beta}$ and $T'^{\gamma\delta}h_{\gamma\delta}$, and calculating the interaction between $T^{\alpha\beta}$ and $T'^{\gamma\delta}$. At the tree level, the interaction between two symmetric conserved sources $T^{\alpha\beta}(x)$ and $T'^{\gamma\delta}(x)$ is

$$\begin{aligned} & \int \frac{d^4k}{(2\pi)^4} \tilde{T}^{\alpha\beta}(k) G_{\alpha\beta\gamma\delta}^{(0)}(k) \tilde{T}'^{\gamma\delta}(-k) \\ &= \int \frac{d^4k}{(2\pi)^4} \frac{32\pi i}{M_{Pl}^2(k^2 - m^2)} \left(\tilde{T}^{\alpha\beta}(k) \tilde{T}'_{\alpha\beta}(-k) - \frac{1}{3} \tilde{T}^\alpha_\alpha(k) \tilde{T}'^\beta_\beta(-k) \right) \end{aligned} \quad (12)$$

We now calculate the interaction at the one loop order; the corresponding diagrams are shown in Fig. 3. We are interested in the regime $k^2 \gg m^2$.

As the sources are transverse, all terms in the propagator enhanced by m^{-4} and m^{-2} vanish upon contracting with $T^{\alpha\beta}$ or $T'^{\alpha\beta}$, except for one term. This non-vanishing term is of the form $\eta_{\alpha\beta}k_\mu k_\nu$, where the two indices of the Minkowski metric are contracted with the energy-momentum tensor. So, one might think that for the first diagram of Fig. 3, the two propagators in the loop each contribute as m^{-4} , and the two external propagators each contribute as m^{-2} . This would lead to a result of the order m^{-12} , but as in the previous section there are cancellations. Unlike in the previous section, however, these cancellations have nothing to do with the mass-shell condition. By direct calculation one finds that the terms of order m^{-12} and m^{-10} vanish, so one is left with contribution of order m^{-8} from the first diagram of Fig. 3. The second diagram has only one propagator inside the loop, so it is at most of order m^{-8} .

To check that the terms of order m^{-8} do not cancel out, it is sufficient to consider traceless sources,

$$T^\alpha_\alpha = T'^\alpha_\alpha = 0$$

This simplifies algebra considerably (in particular, the second diagram in Fig. 3 does not contribute to the order m^{-8}), and we obtain for the sum of the tree level and one-loop interactions,

$$\begin{aligned} & \int \frac{d^4k}{(2\pi)^4} \tilde{T}^{\alpha\beta}(k) \left[G_{\alpha\beta\gamma\delta}^{(0)}(k) + G_{\alpha\beta\gamma\delta}^{(1)}(k) \right] \tilde{T}'^{\gamma\delta}(-k) \\ &= \int \frac{d^4k}{(2\pi)^4} \tilde{T}^{\alpha\beta}(k) \tilde{T}'_{\alpha\beta}(-k) \frac{32\pi i}{M_{Pl}^2(k^2 - m^2)} \\ & \quad \left(1 + \frac{k^{10}}{2160\pi M_{Pl}^2 m^8} \log(k^2) + \frac{P(k)}{M_{Pl}^2 m^8} + \frac{\mathcal{O}(m^{-6})}{M_{Pl}^2} \right) \end{aligned} \quad (13)$$

where $P(k)$ is a polynomial in k .

Thus, the correction of order m^{-8} does not cancel out in the graviton propagator, even for traceless sources. This correction becomes comparable to the tree level term at the energy scale Λ , Eq. (1). This again demonstrates that Λ is indeed the strong coupling scale in massive gravity.

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A Vertices

To obtain the expression for the Einstein action (3) at the third and fourth order in $h_{\mu\nu}$, one makes use the following expansions,

$$\begin{aligned}\sqrt{-g} &= 1 + \frac{1}{2}h^\alpha{}_\alpha + \frac{1}{8}(h^\alpha{}_\alpha)^2 - \frac{1}{4}h_{\alpha\beta}h^{\alpha\beta} + \frac{1}{48}(h^\alpha{}_\alpha)^3 \\ &\quad - \frac{1}{8}h^\alpha{}_\alpha h_{\beta\gamma}h^{\beta\gamma} + \frac{1}{6}h_{\alpha\beta}h^{\beta\gamma}h_\gamma{}^\alpha + \mathcal{O}(h_{\alpha\beta}^4)\end{aligned}\quad (14)$$

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} + h^\mu{}_\sigma h^{\sigma\nu} - h^{\mu\sigma}h_{\sigma\tau}h^{\tau\nu} + \mathcal{O}(h_{\mu\nu}^4)\quad (15)$$

With these, one finds the expressions for the three and four point vertices (cf. Ref. [15]):

$$\begin{aligned}V_3^{(\mu\nu)(\sigma\tau)(\rho\lambda)} &= -\frac{iM_{Pl}^2}{16\pi}\text{Sym}\left\{-\frac{1}{4}P_3\left[\eta^{\mu\nu}\eta^{\sigma\tau}\eta^{\rho\lambda}k_1\cdot k_2\right]\right. \\ &\quad +\frac{1}{4}P_3\left[\eta^{\mu\nu}\eta^{\sigma\rho}\eta^{\tau\lambda}(k_2\cdot k_3 - 2k_1\cdot k_1)\right] \\ &\quad -P_3\left[\eta^{\lambda\mu}\eta^{\nu\sigma}\eta^{\tau\rho}k_1\cdot k_2\right] - \frac{1}{4}P_6\left[\eta^{\mu\nu}\eta^{\sigma\tau}k_1^\rho k_1^\lambda\right] \\ &\quad +P_3\left[\eta^{\mu\nu}\eta^{\sigma\rho}(k_1^\tau k_1^\lambda - \frac{1}{2}k_3^\tau k_2^\lambda)\right] + \frac{1}{2}P_6\left[\eta^{\mu\sigma}\eta^{\nu\tau}k_1^\rho k_1^\lambda\right] \\ &\quad +\frac{1}{2}P_3\left[\eta^{\mu\sigma}\eta^{\nu\tau}k_1^\rho k_2^\lambda\right] + P_3\left[\eta^{\mu\sigma}\eta^{\nu\rho}k_3^\tau k_2^\lambda\right] \\ &\quad \left.+P_6\left[\eta^{\mu\sigma}\eta^{\nu\rho}k_1^\tau k_2^\lambda\right]\right\}\end{aligned}\quad (16)$$

$$\begin{aligned}V_4^{*(\mu\nu)(\sigma\tau)(\rho\lambda)(\pi\xi)} &= -\frac{iM_{Pl}^2}{16\pi}\text{Sym}\left\{\frac{1}{4}P_3\left[\eta^{\mu\sigma}\eta^{\nu\tau}\eta^{\rho\pi}\eta^{\lambda\xi}(k_1\cdot k_2 + 2m^2)\right]\right. \\ &\quad -2P_3\left[\eta^{\xi\mu}\eta^{\nu\sigma}\eta^{\tau\rho}\eta^{\lambda\pi}m^2\right] - \frac{1}{2}P_{12}\left[\eta^{\xi\sigma}\eta^{\tau\rho}\eta^{\lambda\pi}k_2^\mu k_2^\nu\right] \\ &\quad -\frac{1}{2}P_{12}\left[\eta^{\mu\sigma}\eta^{\nu\tau}\eta^{\rho\pi}k_1^\lambda k_1^\xi\right] \\ &\quad \left.+P_{12}\left[\eta^{\sigma\rho}\eta^{\lambda\pi}\eta^{\xi\mu}(k_2^\nu k_1^\tau - k_3^\nu k_4^\tau)\right]\right\}\end{aligned}\quad (17)$$

where $k_1 + k_2 + k_3 = 0$ for the three point vertex and $k_1 + k_2 + k_3 + k_4 = 0$ for the four point vertex. The symbol P means that one has to sum over all distinct permutations of the triplets of indices $1\mu\nu$, $2\sigma\tau$, $3\rho\lambda$ and $4\pi\xi$. The subscript of P indicates the number of distinct permutations over which the summation has to be carried out. The “Sym” symbol means that the total expression has to be symmetrized in each pair of indices $(\mu\nu)$, $(\sigma\tau)$, $(\rho\lambda)$ and $(\pi\xi)$. In fact, this symmetrization is not required as long as one only multiplies the vertices by the propagator (7) or the polarisation tensors (5) which are already symmetric.

The * symbol indicates that, for the four point vertex (17), the four legs are on mass shell. The general off-shell expression of the four point vertex is given in Ref. [15].

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